Math 20550 - Summer 2016
Fundamental Theorem of Line Integrals and Green's Theorem
July 11, 2016

Problem 1. *Evaluate* $\int_C (x^2+y^2)dx + 2xy\ dy$ *where* C *is given by*

(*a*) $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$
(*b*) $x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$

Problem 2. *Is the vector field* $\mathbf{F} = \langle y, -x\rangle$ *conservative? If so, find a potential function.*

Problem 3. *Compute the line integral*

$$\int_C (y^2z + 2xz^2)dx + 2xyz\ dy + (xy^2 + 2x^2z)dz$$

where C *is the path given by* $\mathbf{r}(t) = \left\langle \sqrt{t}, t+1, t^2\right\rangle$, $0 \le t \le 1$.

Problem 4. *Find the work done by the force field* $\mathbf{F}(x,y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$ *in moving a particle from the point* $P = (1,1)$ *to the point* $Q = (2,4)$.

Problem 5. *Is there a vector field* $\mathbf{G}$ *such that*

$$\text{curl } \mathbf{G} = \left\langle xyz, -y^2z, yz^2\right\rangle?$$

Problem 6. *Compute the line integral* $\oint_C xy\ dx + x^2y^3\ dy$ *where* C *is the triangle with vertices* $(0,0)$, $(1,0)$, $(1,2)$.

Problem 7. *Let* D *be the region between the curves* $x^2+y^2 = 4$ *and* $x^2+y^2 = 9$. *Let* C *be the boundary of the region* D.

(*a*) *What does it mean for* C *to be positively oriented? Sketch* C *with its positive orientation.*
(*b*) *Compute the integral* $\int_C (1-y^3)dx + (x^3 + e^{y^2})dy$ *where* C *has positive orientation.*

Problem 8. *Use Green's Theorem to find the area under one arch of the cycloid* $x = t - \sin t$, $y = 1 - \cos t$.

Problem 9. *Compute the line integral* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = \langle xy, y^2\rangle$ *and* C *consists of the line segment from* $(0,0)$ *to* $(0,4)$ *and the piece of the parabola* $y = 4 - x^2$ *from* $(0,4)$ *to* $(1,3)$, *and the line segment from* $(1,3)$ *to* $(1,0)$.

Problem 10. *Suppose we have two objects of masses m and M. Assume the mass M is at the origin. The force due to gravity at the point (x, y, z) is given by*

$$\mathbf{F}(\mathbf{r}) = -\frac{mMG\mathbf{r}}{|\mathbf{r}|^3},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Suppose that P_1 is a distance d_1 from the origin and P_2 is a distance d_2 from the origin.

(a) *Find the work done by $\mathbf{F}$ in moving the mass m from P_1 to P_2 in terms of d_1 and d_2.*

(b) *Find the work done by the gravitational field as the Earth moves from aphelion (its maximum distance of $1.52 \cdot 10^8 km$) to perihelion (its minimum distance of $1.47 \cdot 10^8 km$). You can leave your answers in terms of m, M, and G, or you can use $m = 5.97 \cdot 10^{24} kg$, $M = 1.99 \cdot 10^{30} kg$, and $G = 6.67 \cdot 10^{-11} N \cdot m^2/kg^2$.*